Methods and Applications

Epidemic Threshold Analysis of an SEIMQR Model Incorporating Healthcare Resource Constraints

Qi Sun[1]; Yu Chang[1,#]

ABSTRACT

Introduction: Epidemic models commonly assume unlimited healthcare resources and overlook a critical operational reality: healthcare system overload reduces recovery rates and amplifies outbreak severity. We propose an enhanced susceptible-exposed-infected-mutant-quarantined-recovered (SEIMQR) model that explicitly links the recovery process to time-varying hospital-bed availability, and thereby better reflects practical healthcare constraints.

Methods: The model integrates a saturating, resource-dependent recovery function and uses the quarantine rate as the primary control parameter. We performed a systematic dynamic analysis using bifurcation theory and validated the model through numerical simulations and empirical coronavirus disease 2019 (COVID-19) surveillance data from the United Kingdom. These data covered distinct phases of the pandemic.

Results: Our analysis identified a precise quarantine threshold (δ = 0.573595) that distinguishes disease elimination from endemic persistence. Simulation results show that strengthening quarantine measures effectively suppresses the epidemic scale. Furthermore, the model fit the reported cumulative case data with high accuracy across different variant-dominant periods and policy regimes, which demonstrates its adaptability.

Conclusion: These findings quantify a critical intervention threshold and illustrate the synergistic relationship between medical capacity and quarantine intensity. This study provides a theoretically grounded, resource-aware framework for guiding integrated public health strategies when healthcare resources are finite. Therefore, it supports more realistic epidemic preparedness and response planning.

The global spread of coronavirus disease 2019 (COVID-19) has placed substantial pressure on public health infrastructure worldwide, underscoring the need for epidemiological models that account for real operational limitations in healthcare systems. Established compartmental models, such as susceptible-exposed-infected-recovered (SEIR), provide a fundamental understanding of transmission dynamics. However, they often assume invariant recovery rates, thereby failing to capture the finite, saturable nature of clinical care resources. In practice, caseload surges exceeding healthcare capacity degrade treatment efficacy, prolong infectious periods, and exacerbate the population disease burden. This creates a nonlinear interaction that must be explicitly incorporated into modeling frameworks. To address these complexities, researchers have developed extended modeling architectures that integrate vaccination and quarantine interventions (*1*), distinguish between hospitalization and community care pathways (*2*), and incorporate behavioral dynamics or fractional-order operators (*3*). Advancing this methodological path, the susceptible-exposed-infected-mutant-quarantined-recovered (SEIMQR) model introduces distinct compartments for variant-specific infections and quarantined individuals, offering a refined structure for simulating transmission dynamics under conditions of viral evolution and targeted intervention (*4*).

A common limitation of existing models is their treatment of recovery rates as static parameters, which may not fully capture the dynamic interaction with concurrent healthcare resource demands. This assumption is particularly restrictive during surge conditions, when system capacity constraints are most acute. To address this gap, this study developed an enhanced SEIMQR framework that incorporates a resource-dependent recovery function, dynamically linking treatment outcomes to available hospital capacity.

The study applies dynamic systems theory and bifurcation analysis, using the quarantine rate as a key bifurcation parameter to investigate how finite healthcare resources, in conjunction with intervention policies, shape epidemic trajectories. Our analytical approach combines equilibrium stability analysis with numerical validation using longitudinal COVID-19 surveillance data from the United Kingdom. These data encompass distinct pandemic phases marked by dominant variants and evolving policy regimes. By explicitly embedding healthcare resource constraints into the epidemic model, this study aims to advance a more operationally valid framework for informing public health decision-making under conditions of resource scarcity.

METHODS

Model Formulation

The SEIMQR model extends the classical SEIR framework by incorporating compartments for mutant-strain infections and quarantined populations, a structure that fits empirical COVID-19 data better than simpler models. Building on the framework established by Yu et al. (4), this model effectively captures variant-driven dynamics and basic intervention effects, providing a solid foundation for further development.

Our refinement addresses a key operational constraint: the dynamic relationship between healthcare system load and patient outcomes. Although the original model effectively captures variant dynamics and intervention effects, it, like many compartmental approaches, treats the recovery rate as a constant parameter, independent of the scale of infections. This formulation does not account for the degradation of treatment efficacy that occurs when healthcare systems approach or exceed their operational capacity. In practice, finite resources, such as hospital beds, staffing, and equipment impose an upper limit on medical throughput. As the infected population grows, the per-capita recovery rate declines, becoming state-dependent rather than constant. The absence of this feedback mechanism limits the model's ability to reflect how medical capacity influences equilibrium states and epidemic thresholds.

To capture this saturation effect, we formulate a resource-dependent recovery function that characterizes the dynamic relationship between healthcare capacity and disease progression. The function takes the form of (5):

$$\varepsilon(b, I) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\frac{b}{I+b} \tag{1}$$

where b denotes the normalized number of hospital beds (serving as a proxy for healthcare resources), I is the symptomatic infected fraction, and ε_1 and ε_0 are the attainable recovery rates under sufficient resources and extreme overloads, respectively.

This functional form captures the saturation of medical capacity as the infected population grows, thereby introducing state-dependent feedback into the recovery process. By integrating this mechanism and treating the quarantine rate as a bifurcation parameter, the modified model enables systematic investigation of how finite healthcare resources reshape the system's dynamic landscape, offering a more realistic foundation for assessing public health strategies under resource-constrained scenarios. A dynamic flow diagram of the modified SEIMQR model is shown in Figure 1.

This enhancement yields the modified SEIMQR system:

$$\begin{cases}\dfrac{dS}{dt} = \mu - \alpha S(I+E) - \eta SM - \mu S\\[4pt] \dfrac{dE}{dt} = \alpha S(I+E) + \eta SM - \beta E - \delta E - \mu E\\[4pt] \dfrac{dI}{dt} = \dfrac{1}{2}\beta E - \gamma I - \delta I - \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right)I - \mu I\\[4pt] \dfrac{dM}{dt} = \dfrac{1}{2}\beta E + \gamma I - \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right)M - \delta M - \mu M\\[4pt] \dfrac{dQ}{dt} = \delta(E+I+M) - \mu Q\\[4pt] \dfrac{dR}{dt} = \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right)I\\ \quad + \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right)M - \mu R\end{cases} \tag{2}$$

where $S(t)$, $E(t)$, $I(t)$, $M(t)$, $Q(t)$, $R(t)$ denote the proportions of the susceptible, exposed, symptomatic infected, mutant-strain infected, quarantined, and recovered populations, respectively, satisfying the normalization constraint $S+E+I+M+Q+R=1$. The parameter μ denotes the natural birth and death rate. The coefficients $\alpha$and η represent the transmission probabilities associated with contacts between susceptible individuals and the symptomatic infected/exposed classes and mutant-strain infectives, respectively. The parameter β governs the transition from exposed to infectives, while δ measures the quarantine intensity applied to exposed individuals. The mutation probability γ characterizes the conversion from the symptomatic infected class to the mutant-strain infected class.

Let $X(t) = (E, I, M)^T$. Then, we have $\dfrac{dX}{dt} = F - V$, where

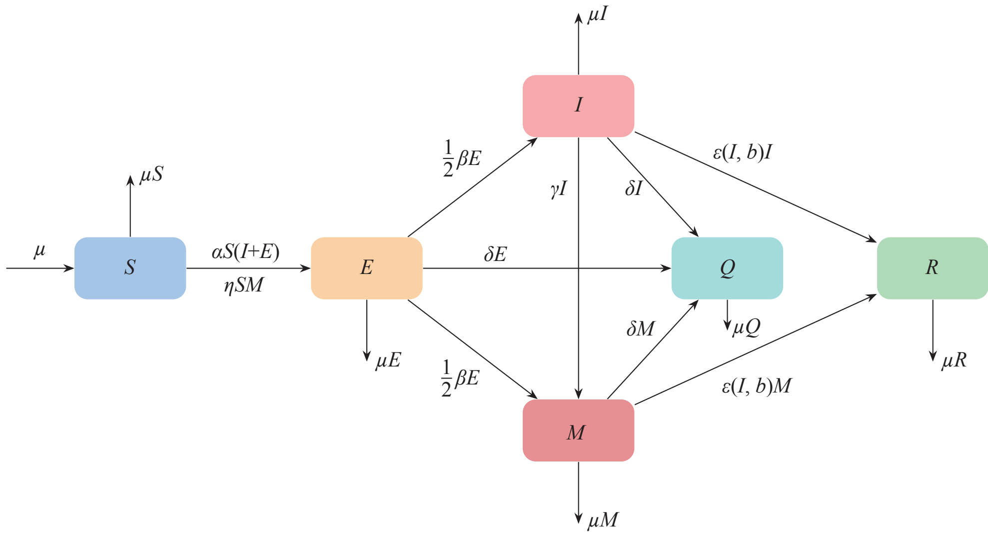

FIGURE 1. State-transition diagram of the modified SEIMQR model.
Abbreviation: SEIMQR=susceptible-exposed-infected-mutant-quarantined-recovered.

$$F = \begin{bmatrix} \alpha S(I+E) + \eta SM \\ 0 \\ 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \beta E + \delta E + \mu E \\ \gamma I + \delta I + \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right) I + \mu I - \dfrac{1}{2}\beta E \\ \left(\varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\dfrac{b}{I+b}\right) M + \delta M + \mu M - \dfrac{1}{2}\beta E - \gamma I \end{bmatrix}. \quad (3)$$

Upon evaluating F and V at the disease-free equilibrium E_0, the corresponding Jacobian matrices are obtained as

$$J_F = \begin{bmatrix} \alpha & \alpha & \eta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$J_V = \begin{bmatrix} \beta+\delta+\mu & 0 & 0 \\ -\dfrac{\beta}{2} & \gamma+\delta+\varepsilon_1+\mu & 0 \\ -\dfrac{\beta}{2} & -\gamma & \delta+\varepsilon_1+\mu \end{bmatrix}. \quad (4)$$

The basic reproduction number R_0 is defined as the spectral radius of the next-generation matrix $J_F J_V^{-1}$, evaluated at the disease-free equilibrium E_0. Therefore, R_0 is given by

$$\rho\left(J_F J_V^{-1}\right) = \frac{\alpha}{\beta+\delta+\mu} + \frac{\alpha\beta}{2(\beta+\delta+\mu)(\gamma+\delta+\mu+\varepsilon_1)} + \frac{\eta\beta(2\gamma+\delta+\mu+\varepsilon_1)}{2(\beta+\delta+\mu)(\gamma+\delta+\mu+\varepsilon_1)(\delta+\mu+\varepsilon_1)} \quad (5)$$

where $\rho(\cdot)$ denotes the spectral radius.

Parameter Estimation

The particle swarm optimization (PSO) algorithm was employed to explore the parameter space and identify the parameters set that minimized the discrepancy between the model outputs and the observed data. In this study, the cumulative number of confirmed cases served as the fitting target, and goodness-of-fit was quantified using the root mean square error (RMSE) as the objective function. Owing to its efficient convergence and robust global search capability, PSO is particularly well-suited for parameter estimation in nonlinear dynamical systems with multiple parameters.

Data Source

COVID-19 cumulative case data were obtained from a worldwide dataset (*6*). The data include daily cumulative confirmed cases for the United Kingdom across three distinct pandemic phases: Phase 1 (Delta-dominant, June 1 to July 15, 2021), Phase 2 (emergence of the Omicron variant, December 20, 2021, to January 20, 2022), and Phase 3 ("living with the virus" policy, April 15 to May 31, 2022). Based on the same data source, the total population of the United Kingdom in 2021 was 67,326,569.

RESULTS

We performed an integrated theoretical and numerical analysis of how the quarantine rate influences epidemic outcomes within the modified SEIMQR model. The quarantine rate was selected as the primary control parameter due to system complexity.

The model consistently admitted a disease-free equilibrium (DFE) under all parameter configurations. However, its practical attainability depends critically on the quarantine rate. When the quarantine rate is low, sustained transmission leads to infected population persisting at nonzero levels, a pattern characteristic of an endemic state. As the rate increases, a critical threshold is reached beyond which sustained transmission cannot be maintained.

Numerical simulations were conducted with the following fixed values: $\mu = 0.1, \alpha = 0.82, \eta = 0.7, \beta = 0.3, \gamma = 0.15, \varepsilon_1 = 0.8, \varepsilon_0 = 0.01$, $b = 0.5$, while the quarantine rate was systematically varied.

Figure 2 shows the system's bifurcation behavior with respect to the quarantine rate δ. For $0 < \delta < 0.573595$, the disease-free state is unstable and a stable endemic state exists, with infected compartments converging to nonzero steady-state levels. At the transcritical bifurcation (TB) value $\delta = 0.573595$, the endemic and disease-free states coincide, marking a threshold transition. For $0.573595 < \delta < 1$, the endemic state disappears and the disease-free state becomes stable, indicating effective epidemic control.

Figure 3 illustrates the synergistic effect of healthcare capacity b and quarantine rate δ on the equilibrium infected proportion I^*. The results confirm that although b does not influence R_0 or the transcritical bifurcation threshold, it strongly modulates the endemic infection level once δ falls below the threshold; this effect becomes more pronounced as the quarantine weakens. In the extinction regime ($0.573595 < \delta < 1$ and $R_0 < 1$), I^* remains at zero and is insensitive to b. In the endemic regime ($0 < \delta < 0.573595$ and $R_0 > 1$), I^* becomes highly responsive to b, with a marked increase in sensitivity as δ decreased.

For empirical validation, we estimated time-varying parameters using particle swarm optimization, fitting COVID-19 cumulative case data from the United Kingdom across three distinct pandemic phases and

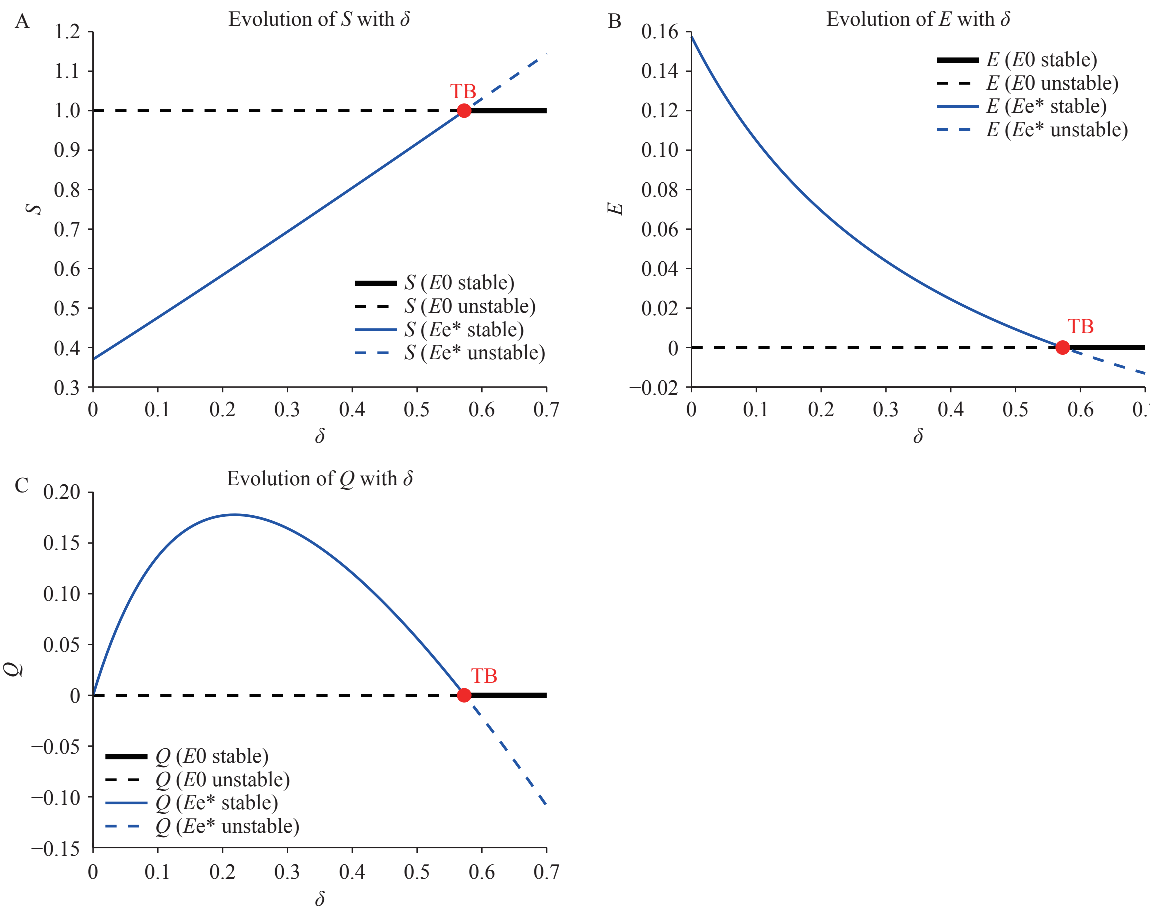

FIGURE 2. Bifurcation diagrams. (A) The quarantine rate δ — Susceptible (S); (B) The quarantine rate δ — Exposed (E); (C) The quarantine rate δ — Quarantined (Q).
Abbreviation: TB=transcritical bifurcation.

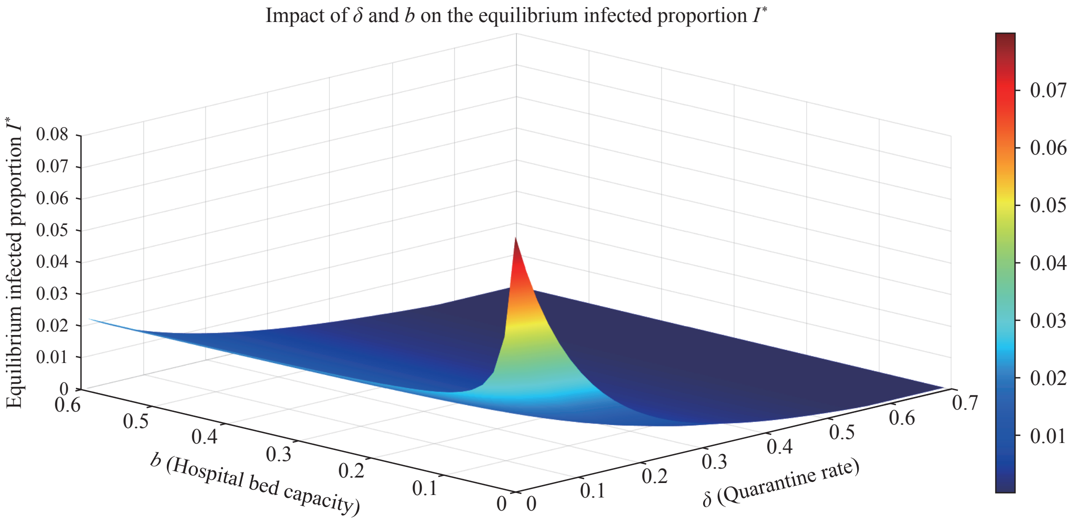

FIGURE 3. Impact of b and δ on the equilibrium infected proportion I^*.

drawing on the organisation for economic co-operation and development (OECD) report *Health at a Glance: Europe 2024* (*7*). Figure 4 shows the fitted cumulative case curves alongside reported data. During the Delta-variant-dominant phase (June 1–July 15, 2021), the model achieved a mean absolute percentage error (MAPE) of 0.11%. During the Omicron emergence period (December 20, 2021–January 20, 2022), the estimated mutant strain transmission rate increased to 0.64, while the quarantine rate decreased to 0.034, successfully capturing the variant's heightened transmissibility and reduced controllability. Under the "living with the virus" policy (April 15–May 31, 2022), the quarantine rate further declined to 0.0085, and the model effectively reproduced the associated increase in infection risk with a MAPE of 0.05%. The fitted parameters, initial conditions, and corresponding basic reproduction numbers R_0 are summarized in Table 1; the coefficients of determination for Figure 4A–C were $R^2 = 0.9993$, $R^2 = 0.9989$ and $R^2 = 0.9913$, respectively. These results confirm that the modified SEIMQR framework can be adapted to diverse epidemiological contexts and policy regimes, accurately capturing shifts in transmission dynamics driven by emerging variants and changes in intervention intensity.

DISCUSSION

This study presents an enhanced SEIMQR modeling framework that explicitly incorporates healthcare capacity constraints, addressing notable limitations of conventional compartmental models. A dynamic, resource-dependent recovery function, $\varepsilon(b, I)$ captures the impact of healthcare system strain on individual patient outcomes and population-level epidemic dynamics. This function $\varepsilon(b, I)$ links the recovery rate to the infected population size: when infections are few, healthcare resources are abundant and the recovery rate approaches its upper bound; as infection grows and the system saturates, the recovery rate declines, prolonging infectious periods and amplifying transmission. Traditional fixed-recovery-rate models cannot capture this feedback.

Theoretical analysis revealed that near the transcritical bifurcation, the system exhibits extreme sensitivity to initial conditions. Minimal differences in the initial infected proportion or model parameters can be amplified by nonlinear mechanisms, causing distinctly different epidemic pathways. From a nonlinear dynamics perspective, this divergence under critical conditions is directly linked to the bifurcation-induced topological change in system stability. This underscores the pivotal regulatory role of the quarantine rate in epidemic transmission dynamics.

Validation against longitudinal COVID-19 data from the United Kingdom demonstrated the model's adaptability across diverse epidemiological contexts, including periods dominated by distinct variants and varying policy interventions. These results highlight the importance of integrating empirical healthcare resource limitations into epidemic forecasting, especially when healthcare system saturation can delay control efforts and prolong community transmission.

Consistent with our findings, identifying a precise

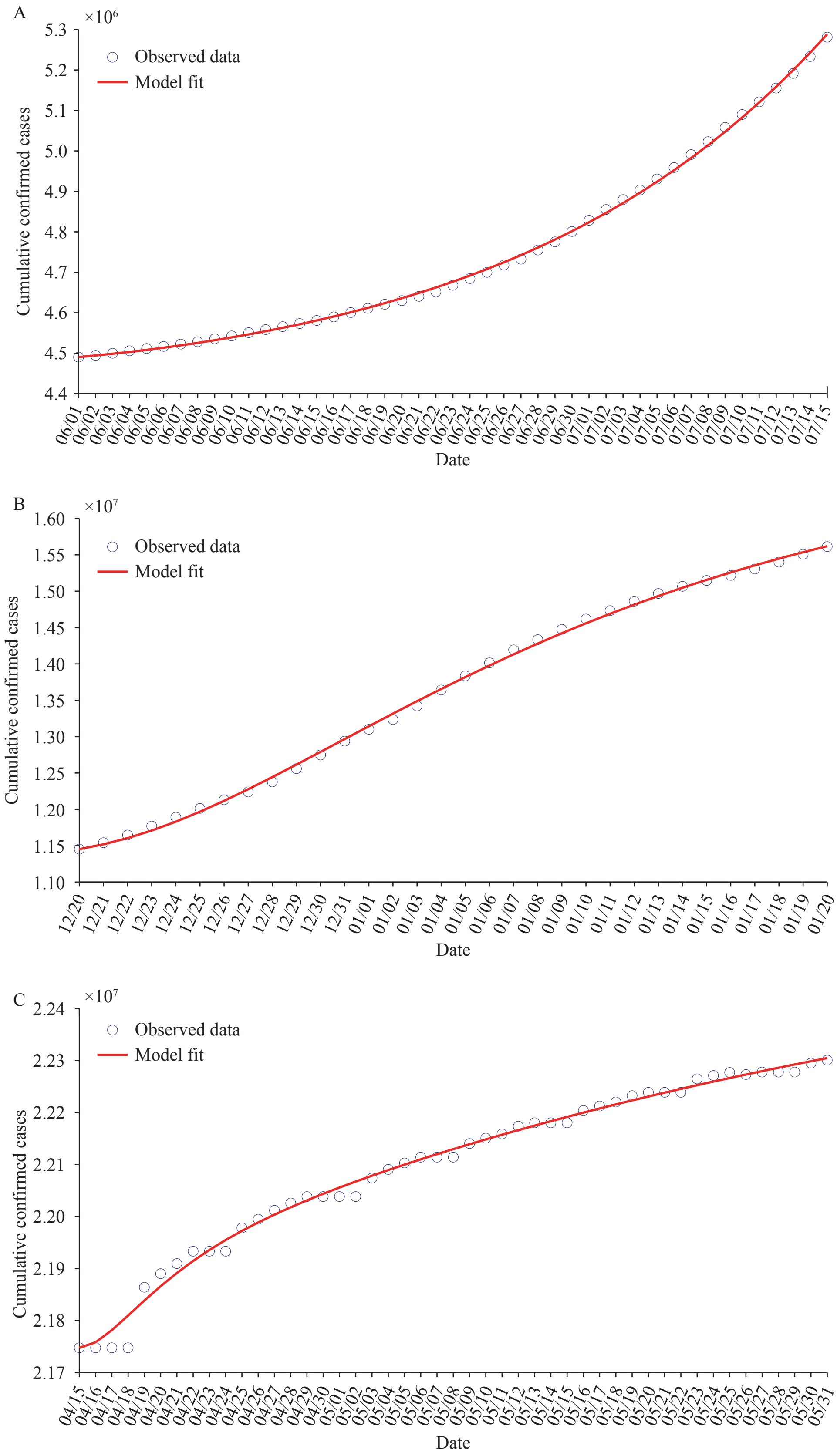

FIGURE 4. Model-fitted cumulative confirmed COVID-19 cases in the United Kingdom across three representative periods. (A) 1 June–15 July 2021, (B) 20 December 2021–20 January 2022, and (C) 15 April–31 May 2022.
Abbreviation: COVID-19=coronavirus disease 2019.

TABLE 1. Model parameters, R_0 and initial values across three distinct pandemic phases in the United Kingdom.

Phase	Parameter	Value	Source	Parameter	Value	Source
	μ	0.00001094	(6)	γ	0.10158413	Fitting
	α	0.12530421	Fitting	ε_1	0.01398170	Fitting
	η	0.26454923	Fitting	ε_0	0.01051092	Fitting
1	β	0.67564826	Fitting	b	0.00225051	Fitting
	δ	0.11270636	Fitting	R_0	1.68704179	Calculated
	$S(0)$	$E(0)$	$I(0)$	$M(0)$	$Q(0)$	$R(0)$
	0.88649523	0.00008944	0.00051128	0.00000094	0.00000094	0.11290217
	μ	0.00001094	(6)	γ	0.03684764	Fitting
	α	0.05825147	Fitting	ε_1	0.09273017	Fitting
	η	0.64340215	Fitting	ε_0	0.04368076	Fitting
2	β	0.45728898	Fitting	b	0.00200918	Fitting
	δ	0.03369863	Fitting	R_0	3.18912013	Calculated
	$S(0)$	$E(0)$	$I(0)$	$M(0)$	$Q(0)$	$R(0)$
	0.13935162	0.00188082	0.17271673	0.00000036	0.33032572	0.35572475
	μ	0.00001094	(6)	γ	0.25964701	Fitting
	α	0.65734363	Fitting	ε_1	0.01203041	Fitting
	η	0.15883262	Fitting	ε_0	0.00846606	Fitting
3	β	0.50308248	Fitting	b	0.00192497	Fitting
	δ	0.00847932	Fitting	R_0	9.77151472	Calculated
	$S(0)$	$E(0)$	$I(0)$	$M(0)$	$Q(0)$	$R(0)$
	0.03477658	0.00000047	0.03397416	0.00166396	0.46440994	0.46517489

Note: $S(0)$ is the initial proportion of the susceptible population. $E(0)$ is the initial proportion of the exposed population. $I(0)$ is the initial proportion of the symptomatic infected population. $M(0)$ is the initial proportion of the mutant-strain infected population. $Q(0)$ is the initial proportion of the quarantined population. $R(0)$ is the initial proportion of the recovered population. R_0 is the basic reproduction number.

quarantine threshold provides evidence-based guidance for containment strategies in resource-limited settings. The transcritical bifurcation value serves as a clear operational target; control measures should be dynamically calibrated to sustain it. Where such intensity is unattainable, our analysis shows that expanding healthcare capacity b is essential to mitigate the long-term infection burden. Furthermore, the model elucidates key synergies between healthcare resource availability and quarantine effectiveness. As healthcare systems approach saturation, strengthening quarantine alone may yield diminishing returns. Coordinated interventions combining temporary healthcare expansion with targeted quarantine offer superior outcomes compared with single-component approaches.

Although this model advances the understanding of resource-constrained epidemic dynamics, it has several limitations. The current framework does not account for spatial heterogeneity in transmission and healthcare access, dynamic behavioral adaptations, or the evolving effects of vaccination and population immunity. Future research should incorporate these elements to develop more comprehensive models for complex, practical public health challenges.

Collectively, this study establishes a practical resource-aware modeling framework directly linking healthcare constraints to intervention planning. By providing decision-makers a more realistic scenario-evaluation tool, it contributes to improved epidemic preparedness and response in resource-limited environments.

Conflicts of interest: No conflicts of interest.

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Corresponding author: Yu Chang, changyu@mail.buct.edu.cn.

1 College of Mathematics and Physics, Beijing University of Chemical Technology, Beijing, China.

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REFERENCES

1. Annas S, Pratama MI, Rifandi M, Sanusi W, Side S. Stability analysis and numerical simulation of SEIR model for pandemic COVID-19 spread in Indonesia. Chaos Solitons Fractals 2020;139:110072. https://doi.org/10.1016/j.chaos.2020.110072.
2. He SB, Peng YX, Sun KH. SEIR modeling of the COVID-19 and its dynamics. Nonlinear Dyn 2020;101(3):1667 – 80. https://doi.org/10.1007/s11071-020-05743-y.
3. Joshi H, Yavuz M, Townley S, Jha BK. Stability analysis of a non-singular fractional-order covid-19 model with nonlinear incidence and treatment rate. Phys Scr 2023;98(4):045216. https://doi.org/10.1088/1402-4896/acbe7a.
4. Yu ZH, Zhang JM, Zhang Y, Cong XY, Li XB, Mostafa AM. Mathematical modeling and simulation for COVID-19 with mutant and quarantined strategy. Chaos Solitons Fractals 2024;181:114656. https://doi.org/10.1016/j.chaos.2024.114656.
5. Shan CH, Zhu HP. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. J Differ Equations 2014;257(5):1662 – 88. https://doi.org/10.1016/j.jde.2014.05.030.
6. Mathieu E, Ritchie H, Rodés-Guirao L, Appel C, Gavrilov D, Giattino C, et al. COVID-19 pandemic. 2020. https://ourworldindata.org/coronavirus. [2024-7-18].
7. OECD. Health at a glance: Europe 2024: state of health in the EU cycle. OECD Publishing. 2024. http://dx.doi.org/10.1787/b3704e14-en. [2024-11-18].